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Differences between charged-current coefficient functions

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Abstract

Second- and third-order results are presented for the structure functions of charged-current deep-inelastic scattering in the framework of massless perturbative QCD. We write down the two-loop differences between the corresponding crossing-even and -odd coefficient functions, including those for the longitudinal structure function not covered in the literature so far. At three loops we compute the lowest five moments of these differences for all three structure functions and provide approximate expressions in Bjorken-x space. Also calculated is the related third-order coefficient-function correction to the Gottfried sum rule. We confirm the conjectured suppression of these quantities if the number of colours is large. Finally we derive the second- and third-order QCD contributions to the Paschos-Wolfenstein ratio used for the determination of the weak mixing angle from neutrino-nucleon deep-inelastic scattering. These contributions are found to be small.

1 Introduction

Structure functions in deep-inelastic scattering (DIS) are among the most extensively measured observables. Today the combined data from fixed-target experiments and the HERA collider spans about four orders of magnitude in both Bjorken-x and the scale $Q^2 = -q^2$ given by the momentum q of the exchanged electroweak gauge boson [1]. In this article we focus on the W-exchange charged-current (CC) case, see Refs. [2–4] and [5–8] for recent measurements in neutrino DIS and at HERA. With six structure functions, $F_2^{W^\pm}$, $F_3^{W^\pm}$ and $F_L^{W^\pm}$, this case has a far richer structure than, for example, electromagnetic DIS with only two independent observables, F_2 and F_L.

More detailed measurements are required to fully exploit the resulting potential, for instance at a future neutrino factory, see Ref. [9], and the LHeC, the proposed high-luminosity electron-proton collider at the LHC [10]. Already now, however, charged-current DIS provides important information on the parton structure of the proton, e.g., its flavour decomposition and the valence-quark distributions. Moreover, present results are also sensitive to electroweak parameters of the Standard Model such as $\sin^2\theta_W$, see Ref. [11], and the space-like W-boson propagator [12]. As discussed, for example, in Refs. [13–16], a reliable determination of $\sin^2\theta_W$ from neutrino DIS requires a detailed understanding of non-perturbative and perturbative QCD effects.

The perturbative calculations for the unpolarised structure functions in DIS have almost been completed to the next-to-next-to-leading order (NNLO) of massless QCD. These results include the splitting functions, controlling the scale evolution of the parton distributions, to the third order in the strong coupling constant α_s [17, 18], as well as the hard-scattering coefficient functions for F_1, F_2 and F_3 to second order in α_s [19–23]. For the longitudinal structure function $F_L = F_2 - 2xF_1$ the third-order coefficient functions are required at NNLO. So far these quantities have been computed only for electromagnetic (photon-exchange) DIS [24, 25]. In fact, it appears that even the second-order coefficient functions for the charged-current F_L have not been fully presented in the literature.

It is convenient to consider linear combinations of the charged-current structure functions $F_a^{W^\pm}$ with simple properties under crossing, such as $F_a^{\nu p \pm \bar{\nu} p}$ ($a = 2,\, 3,\, L$) for neutrino DIS. For all these combinations either the even or odd moments can be calculated in Mellin-N space in the framework of the operator product expansion (OPE), see Ref. [26]. The results for the third-order coefficient functions for the even-N combinations $F_{2,L}^{\nu p+\bar{\nu} p}$ can be taken over from electromagnetic DIS [24, 25]. Also the coefficient function for the odd-N based quantity $F_3^{\nu p+\bar{\nu} p}$ is completely known at three-loop accuracy, with the results only published via compact parametrizations so far [27]. For the remaining combinations $F_{2,L}^{\nu p-\bar{\nu} p}$ and $F_3^{\nu p-\bar{\nu} p}$, on the other hand, only the first five odd and even integer moments of the respective coefficient functions have been calculated to third order in Ref. [28] following the approach of Refs. [29–31] based on the MINCER program [32, 33].

The complete results of Refs. [24, 25, 27] fix all even and odd moments N. Hence already the present knowledge is sufficient to determine also the lowest five moments of the differences of corresponding even-N and odd-N coefficient functions and to address a theoretical conjecture [34] for these quantities. Furthermore these moments facilitate x-space approximations in the style of, e.g, Ref. [35] which are sufficient for most phenomenological purposes, including the

determination of the third-order QCD corrections to the Paschos-Wolfenstein relation [36] used for the extraction of $\sin^2\theta_W$ from neutrino DIS.

The outline of this article is as follows. In Section 2 we briefly specify our notations and write down the complete second-order results $\delta c_a^{(2)}(x)$ for the above coefficient-function differences. We discuss their behaviour at the end points $x = 0$ and $x = 1$, and provide compact but accurate parametrizations for use in numerical applications. We then proceed, in Section 3, to our new results for the five lowest odd moments of $\delta c_{2,L}^{(3)}$ and even moments of $\delta c_3^{(3)}$, as a byproduct deriving the third-order coefficient-function correction to the Gottfried sum rule. These three-loop moments are presented in a numerical form and employed to construct x-space approximations valid at $x \gtrsim 10^{-2}$. In Section 4 we address the numerical implications of our results. In particular we discuss the higher-order QCD corrections to the Paschos-Wolfenstein relation. Our findings are finally summarized in Section 5. The lengthy full expressions of the new third-order moments in terms of fractions and the Riemann ζ-function can be found in the Appendix.

2 The complete second-order results

We define the even-odd differences of the CC coefficient functions C_a for $a = 2,\, 3,\, L$ as

$$\delta C_{2,L} \;=\; C_{2,L}^{\nu p+\bar{\nu}p} - C_{2,L}^{\nu p-\bar{\nu}p}\;, \qquad \delta C_3 \;=\; C_3^{\nu p-\bar{\nu}p} - C_3^{\nu p+\bar{\nu}p}\;. \tag{2.1}$$

The signs are chosen such that the differences are always 'even – odd' in the moments N accessible by the OPE [26], and it is understood that the $d^{abc}d_{abc}$ part of $C_3^{\nu p+\bar{\nu}p}$ [27, 31] is removed before the difference is formed. The non-singlet quantities (2.1) have an expansion in powers of α_s,

$$\delta C_a \;=\; \sum_{l=2} a_s^l\, \delta c_a^{(l)} \tag{2.2}$$

where, as throughout this and the next section, we are have normalized the expansion parameter as $a_s = \alpha_s/(4\pi)$. There are no first-order contributions to these differences, hence the sums start at $l = 2$ in Eq. (2.2).

All known DIS coefficient functions in massless perturbative QCD can be expressed in terms of the harmonic polylogarithms $H_{m_1,\ldots,m_w}(x)$ with $m_j = 0, \pm 1$. Our notation for these functions follows Ref. [37] to which the reader is referred for a detailed discussion. For $w \leq 3$ the harmonic polylogarithms can be expressed in terms of standard polylogarithms; a complete list can be found in Appendix A of Ref. [23]. A FORTRAN programs for these functions up to weight $w = 4$ has been provided in Ref. [38], with an unpublished extension also covering $w = 5$. In the remainder of this section we employ the short-hand notation

$$H_{\underbrace{0,\ldots,0}_{m},\pm 1,\underbrace{0,\ldots,0}_{n},\pm 1,\ldots}(x) \;=\; H_{\pm(m+1),\pm(n+1),\ldots}(x) \tag{2.3}$$

and additionally suppress the arguments of the harmonic polylogarithms for brevity.

Exact expressions for (moments of) the coefficient functions will be given in terms of the $\mathrm{SU}(N_c)$ colour factors $C_A = N_c$ and $C_F = (N_c^2-1)/(2N_c)$, while we use the QCD values $C_A = 3$ and $C_F = 4/3$ in numerical results. All our results are presented in the $\overline{\mathrm{MS}}$ scheme for the standard choice $\mu_r = \mu_f = Q$ of the renormalization and factorization scales.

The second-order coefficient functions $\delta c_2^{(2)}$ and $\delta c_L^{(2)}$ for the even-odd differences of $F_{2,L}$ read

$$\begin{aligned}
\delta c_2^{(2)}(x) \;=\;& C_F[C_F - C_A/2]\Big(-\frac{324}{5} + 112(1+x)^{-1}\zeta_3 + \frac{16}{5}x^{-1} + \frac{164}{5}x + \frac{144}{5}x^2 \\
& - 40\zeta_3 + 136\zeta_3 x + 8\zeta_2 + 56\zeta_2 x + 96\zeta_2 x^2 - \frac{144}{5}\zeta_2 x^3 - 32\mathrm{H}_{-2,0} \\
& + 96\mathrm{H}_{-2,0}(1+x)^{-1} + 128\mathrm{H}_{-2,0}x - 128\mathrm{H}_{-1}(1+x)^{-1}\zeta_2 + 48\mathrm{H}_{-1}\zeta_2 \\
& - 144\mathrm{H}_{-1}\zeta_2 x + 32\mathrm{H}_{-1,-1,0} - 128\mathrm{H}_{-1,-1,0}(1+x)^{-1} - 224\mathrm{H}_{-1,-1,0}x \\
& + 64\mathrm{H}_{-1,0} + \frac{16}{5}\mathrm{H}_{-1,0}x^{-2} + 64\mathrm{H}_{-1,0}x + 96\mathrm{H}_{-1,0}x^2 - \frac{144}{5}\mathrm{H}_{-1,0}x^3 \\
& - 64\mathrm{H}_{-1,0,0} + 160\mathrm{H}_{-1,0,0}(1+x)^{-1} + 160\mathrm{H}_{-1,0,0}x + 64\mathrm{H}_{-1,2}(1+x)^{-1} \\
& - 32\mathrm{H}_{-1,2} + 32\mathrm{H}_{-1,2}x + \frac{28}{5}\mathrm{H}_0 - 32\mathrm{H}_0(1+x)^{-1} - \frac{16}{5}\mathrm{H}_0 x^{-1} - \frac{292}{5}\mathrm{H}_0 x \\
& + 32\mathrm{H}_0(1+x)^{-1}\zeta_2 + \frac{144}{5}\mathrm{H}_0 x^2 - 16\mathrm{H}_0\zeta_2 + 16\mathrm{H}_0\zeta_2 x - 16\mathrm{H}_{0,0} - 64\mathrm{H}_{0,0}x \\
& - 96\mathrm{H}_{0,0}x^2 + \frac{144}{5}\mathrm{H}_{0,0}x^3 + 24\mathrm{H}_{0,0,0} - 48\mathrm{H}_{0,0,0}(1+x)^{-1} - 24\mathrm{H}_{0,0,0}x \\
& - 32\mathrm{H}_1 + 32\mathrm{H}_1 x - 16\mathrm{H}_2 - 16\mathrm{H}_2 x + 16\mathrm{H}_3 - 32\mathrm{H}_3(1+x)^{-1} - 16\mathrm{H}_3 x\Big)\;, \qquad (2.4)\\
\delta c_L^{(2)}(x) \;=\;& C_F[C_F - C_A/2]\Big(\frac{64}{5}x^{-1} - \frac{416}{5} + \frac{256}{5}x + \frac{96}{5}x^2 + 64\zeta_3 x + 32\zeta_2 x + 64\zeta_2 x^2 \\
& - \frac{96}{5}\zeta_2 x^3 + 64\mathrm{H}_{-2,0}x - 64\mathrm{H}_{-1}\zeta_2 x - 128\mathrm{H}_{-1,-1,0}x + 64\mathrm{H}_{-1,0} + \frac{64}{5}\mathrm{H}_{-1,0}x^{-2} \\
& - 32\mathrm{H}_{-1,0}x^{-1} + 64\mathrm{H}_{-1,0}x + 64\mathrm{H}_{-1,0}x^2 - \frac{96}{5}\mathrm{H}_{-1,0}x^3 + 64\mathrm{H}_{-1,0,0}x + \frac{32}{5}\mathrm{H}_0 \\
& - \frac{64}{5}\mathrm{H}_0 x^{-1} - \frac{448}{5}\mathrm{H}_0 x + \frac{96}{5}\mathrm{H}_0 x^2 - 32\mathrm{H}_{0,0}x - 64\mathrm{H}_{0,0}x^2 + \frac{96}{5}\mathrm{H}_{0,0}x^3\Big)\;. \qquad (2.5)
\end{aligned}$$

The corresponding quantity $\delta c_3^{(2)}$ for the charged-current structure functions F_3 is given by

$$\begin{aligned}
\delta c_3^{(2)}(x) \;=\;& \delta c_2^{(2)}(x) - C_F[C_F - C_A/2]\Big(-\frac{624}{5} + \frac{16}{5}x^{-1} + \frac{464}{5}x + \frac{144}{5}x^2 + 32\zeta_3 \\
& + 96\zeta_3 x - 16\zeta_2 + 48\zeta_2 x + 80\zeta_2 x^2 - \frac{144}{5}\zeta_2 x^3 + 32\mathrm{H}_{-2,0} + 96\mathrm{H}_{-2,0}x \\
& - 32\mathrm{H}_{-1}\zeta_2 - 96\mathrm{H}_{-1}\zeta_2 x - 64\mathrm{H}_{-1,-1,0} - 192\mathrm{H}_{-1,-1,0}x + 64\mathrm{H}_{-1,0} \\
& + \frac{16}{5}\mathrm{H}_{-1,0}x^{-2} - 16\mathrm{H}_{-1,0}x^{-1} + 64\mathrm{H}_{-1,0}x + 80\mathrm{H}_{-1,0}x^2 - \frac{144}{5}\mathrm{H}_{-1,0}x^3 \\
& + 32\mathrm{H}_{-1,0,0} + 96\mathrm{H}_{-1,0,0}x - \frac{16}{5}\mathrm{H}_0 x^{-1} - \frac{112}{5}\mathrm{H}_0 - \frac{592}{5}\mathrm{H}_0 x + \frac{144}{5}\mathrm{H}_0 x^2 \\
& + 16\mathrm{H}_{0,0} - 48\mathrm{H}_{0,0}x - 80\mathrm{H}_{0,0}x^2 + \frac{144}{5}\mathrm{H}_{0,0}x^3\Big)\;. \qquad (2.6)
\end{aligned}$$

Expressions equivalent to Eqs. (2.4) and (2.6) have first been published in Refs. [20] and [22], respectively, and were later confirmed in Ref. [23]. To the best of our knowledge, on the other hand, the function $\delta c_L^{(2)}$ has not been documented in the literature before, see, e.g., Ref. [39] and references therein. It was however calculated by the authors of Refs. [20–22], distributed in a FORTRAN package of the two-loop coefficient functions, and employed for the parametrizations of Ref. [40]. Our expression (2.5) agrees with this unpublished result.

It is instructive to briefly consider the end-point limits of the above results. Suppressing the ubiquitous factor $C_F C_{FA} \equiv C_F [C_F - C_A/2]$, the small-$x$ behaviour of Eqs. (2.4) – (2.6) is

$$\begin{aligned} \delta c_2^{(2)}(x) &\simeq -4\ln^3 x - 8\ln^2 x - (28 - 16\zeta_2)\ln x - 64 + 8\zeta_2 + 72\zeta_3 + \ldots \\ \delta c_3^{(2)}(x) &\simeq -4\ln^3 x - 16\ln^2 x + (12 + 16\zeta_2)\ln x + 44 + 24\zeta_2 + 40\zeta_3 + \ldots \\ \delta c_L^{(2)}(x) &\simeq -32\ln x - 48 + \ldots \; . \end{aligned} \tag{2.7}$$

Thus the even-odd differences are not suppressed with respect to the $\nu p + \bar{\nu} p$ two-loop non-singlet coefficient functions for $x \to 0$: the same powers of $\ln x$ enter Eqs. (2.7) and those quantities. At large x, on the other hand, all three functions $\delta c_a^{(2)}$ are suppressed by factors $(1-x)^2$ times logarithms, reading

$$\begin{aligned} \delta c_2^{(2)}(x) &= -(12 - 8\zeta_2)\,[1-x]\, C_F C_{FA} + O\left([1-x]^2\right) \\ \delta c_3^{(2)}(x) &= (20 - 8\zeta_2)\,[1-x]\, C_F C_{FA} + O\left([1-x]^2\right) \\ \delta c_L^{(2)}(x) &= (32 - 16\zeta_2)\,[1-x]^2\, C_F C_{FA} + O\left([1-x]^3\right) \; . \end{aligned} \tag{2.8}$$

The differences $\delta c_2^{(2)}(x)$ and $\delta c_L^{(2)}(x)$ (both multiplied by -1 for display purposes) are compared to the corresponding even-N $\nu p + \bar{\nu} p$ coefficient functions in Fig. 1. The quantities (2.4) and (2.5) are negligible at $x \gtrsim 0.1$ and at $x \gtrsim 0.3$, respectively, but indeed comparable to the even-moment coefficient functions at small x. The corresponding results for F_3 are qualitative similar to those for F_2, but with $\delta c_3^{(2)}(x)$ small down to $x \simeq 0.01$.

For certain numerical applications, for instance for use with complex-N packages like Ref. [41], it is convenient to have parametrizations of Eqs. (2.4) – (2.6) in terms of elementary functions. With an error of less than 0.1% these functions can be approximated by

$$\begin{aligned} \delta c_2^{(2)}(x) &\simeq \{-9.1587 - 57.70x + 72.29x^2 - 5.689x^3 - xL_0\,(68.804 + 24.40L_0 \\ &\quad + 2.958L_0^2) + 0.249L_0 + 8/9\,L_0^2\,(2 + L_0)\}\,(1-x) \; , \\ \delta c_3^{(2)}(x) &\simeq \{-29.65 + 116.05x - 71.74x^2 - 16.18x^3 + xL_0\,(14.60 + 69.90x \\ &\quad - 0.378L_0^2) - 8.560L_0 + 8/9\,L_0^2\,(4 + L_0)\}\,(1-x) \; , \\ \delta c_L^{(2)}(x) &\simeq \{10.663 - 5.248x - 7.500x^2 + 0.823x^3 + xL_0\,(11.10 + 2.225L_0 \\ &\quad - 0.128L_0^2) + 64/9\,L_0\}\,(1-x)^2 \; . \end{aligned} \tag{2.9}$$

Here we have employed the short-hand $L_0 = \ln x$ and inserted the QCD values of C_F and C_A.

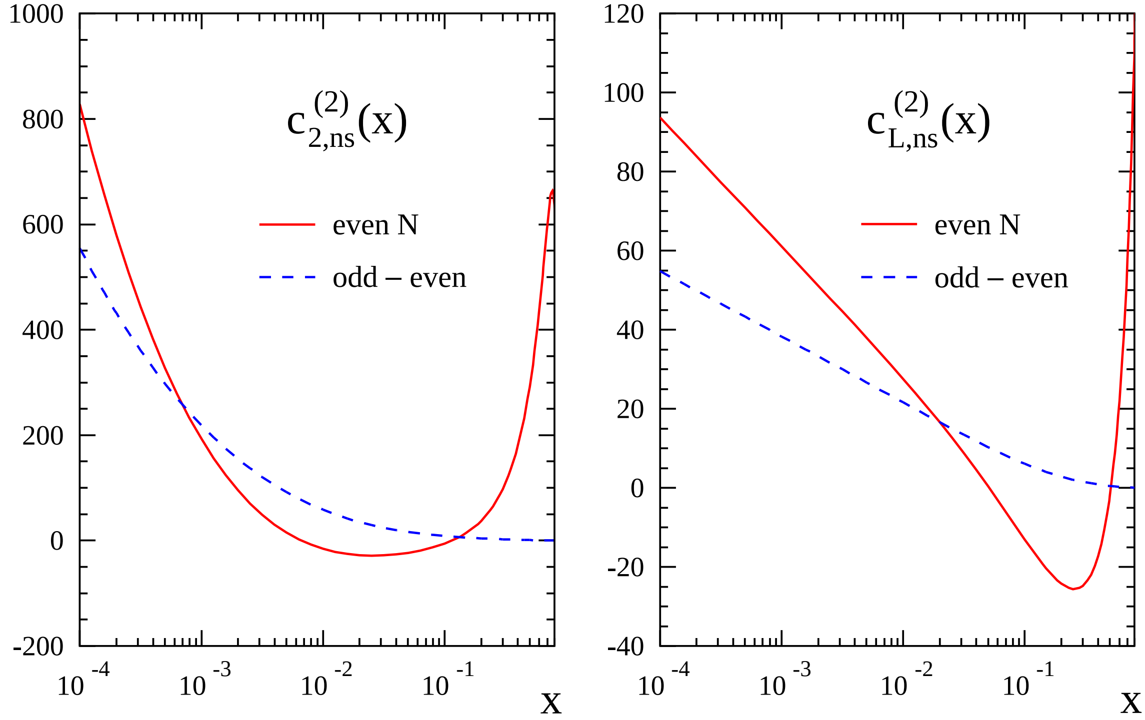

Figure 1: The odd – even non-singlet differences $-\delta c^{(2)}_{2,L}(x)$ of Eqs. (2.4) and (2.5), compared at $x \le 0.8$ to the corresponding even-N coefficient functions calculated in Refs. [19, 20, 23].

3 Third-order moments and approximations

Recently the first five odd-integer moments have been computed of the third-order coefficient functions for $F_{2,L}^{\nu p-\bar{\nu}p}$ in charged-current DIS, together with the corresponding moments $N = 2, \ldots, 10$ for $F_3^{\nu p-\bar{\nu}p}$ [28]. Unlike previous fixed-N calculations, the complete three-loop results for $F_{2,L}^{\nu p+\bar{\nu}p}$ [24, 25][1] and $F_3^{\nu P+\bar{\nu}P}$ [27] facilitate analytic continuations to these values of N. We have performed this continuation using the x-space expressions in terms of harmonic polylogarithms [37] and the Mellin transformation package provided with version 3 of FORM [42]. Thus we are in a position to derive the respective lowest five moments of the hitherto unknown third-order contributions to the even-odd differences (2.1). These moments represent the main new results of this article. With one exception (see below) the exact SU(N_c) expressions are however deferred to the Appendix.

Here we present numerical results for QCD, using the conventions introduced at the beginning of Section 2, recall especially $a_s \equiv \alpha_s/(4\pi)$ and the scale choice $\mu_r = \mu_f = Q$. In addition n_f denotes the number of effectively massless quark flavours, and we use the notation $\delta C_{a,N}$ for the N-th moment of $\delta C_a(x)$. The results for F_2 and F_L read

$$\begin{aligned}
\delta C_{2,1} &= -4.378539253\, a_s^2 + a_s^3\,(-125.2948456 - 0.6502282123\, n_f) \\
\delta C_{2,3} &= -0.138066958\, a_s^2 + a_s^3\,(-5.554493975 + 0.1939792023\, n_f)
\end{aligned}$$

[1] The α_s^3 coefficient functions for this process are those of photon-exchange DIS, but without the contributions of the fl_{11} flavour classes, see Fig. 1 of Ref. [25], where the two photons couple to different quark loops.

$$
\begin{aligned}
\delta C_{2,5} &= -0.032987989\, a_s^2 + a_s^3 \left(-0.707322026 + 0.0004910378\, n_f\right) \\
\delta C_{2,7} &= -0.013235254\, a_s^2 + a_s^3 \left(-0.008816536 - 0.0201069660\, n_f\right) \\
\delta C_{2,9} &= -0.006828983\, a_s^2 + a_s^3 \left(0.133159220 - 0.0200289710\, n_f\right)
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
\delta C_{L,1} &= -2.138954096\, a_s^2 + a_s^3 \left(-106.6667685 + 3.294301343\, n_f\right) \\
\delta C_{L,3} &= -0.078259985\, a_s^2 + a_s^3 \left(-9.239637919 + 0.2718024935\, n_f\right) \\
\delta C_{L,5} &= -0.016892540\, a_s^2 + a_s^3 \left(-2.548566852 + 0.0650677125\, n_f\right) \\
\delta C_{L,7} &= -0.006263113\, a_s^2 + a_s^3 \left(-1.075400460 + 0.0251053847\, n_f\right) \\
\delta C_{L,9} &= -0.003001231\, a_s^2 + a_s^3 \left(-0.560603262 + 0.0122952192\, n_f\right) .
\end{aligned} \tag{3.2}
$$

The lowest even moments for the structure function F_3 are given by

$$
\begin{aligned}
\delta C_{3,2} &= -0.1135841071\, a_s^2 + a_s^3 \left(8.386266870 + 0.0605431788\, n_f\right) \\
\delta C_{3,4} &= -0.0683669250\, a_s^2 + a_s^3 \left(-1.237248886 + 0.0971522112\, n_f\right) \\
\delta C_{3,6} &= -0.0350849853\, a_s^2 + a_s^3 \left(-1.370404531 + 0.0496762716\, n_f\right) \\
\delta C_{3,8} &= -0.0208455457\, a_s^2 + a_s^3 \left(-1.052847874 + 0.0282541123\, n_f\right) \\
\delta C_{3,10} &= -0.0137316528\, a_s^2 + a_s^3 \left(-0.798850682 + 0.0177100327\, n_f\right) .
\end{aligned} \tag{3.3}
$$

The new α_s^3 contributions are rather large if compared to the leading second-order results also included in Eqs. (3.1) – (3.3) with, e.g., $a_s = 1/50$ corresponding to $\alpha_s \simeq 0.25$. Except for the lowest moment for $a = 2, L$, on the other hand, the integer-N differences $\delta C_{a,N}$ are entirely negligible compared to the $\nu p \pm \bar{\nu} p$ moments of Refs. [28, 31].

Before we turn to the x-space implications of Eqs. (3.1) – (3.3), let us briefly discuss some interesting structural features of our third-order results. For this purpose we consider the exact SU(N_c) expression for the lowest moment of $\delta c_2^{(3)}$ given by

$$
\begin{aligned}
\delta c_{2,1}^{(3)} = \; & C_F C_{FA}^2 \left(\frac{175030}{81} - \frac{49216}{27}\zeta_2 + \frac{404720}{81}\zeta_3 - \frac{562784}{135}\zeta_2^2 + \frac{33200}{9}\zeta_2\zeta_3 \right. \\
& \left. \qquad - \frac{4160}{9}\zeta_5 - \frac{8992}{63}\zeta_2^3 - \frac{1472}{3}\zeta_3^2 \right) \\
& + C_F^2 C_{FA} \left(-\frac{303377}{162} + \frac{41350}{27}\zeta_2 - \frac{363896}{81}\zeta_3 + \frac{396824}{135}\zeta_2^2 - \frac{26000}{9}\zeta_2\zeta_3 \right. \\
& \left. \qquad + \frac{25616}{9}\zeta_5 + \frac{1456}{3}\zeta_3^2 - \frac{56432}{315}\zeta_2^3 \right) \\
& + C_F C_{FA} n_f \left(\frac{8786}{81} - \frac{3056}{27}\zeta_2 + \frac{39592}{81}\zeta_3 + \frac{1408}{9}\zeta_2\zeta_3 - \frac{30424}{135}\zeta_2^2 - \frac{1792}{9}\zeta_5 \right) .
\end{aligned} \tag{3.4}
$$

As all other calculated moments of the functions $\delta c_a^{(2)}(x)$, this result contains an overall factor $C_{FA} = C_F - C_A/2 = -1/(2N_c)$. Hence the third-order even-odd differences are suppressed in the

large-N_c limit as conjectured, to all orders, in Ref. [34] on the basis of two-loop results in particular for $N=1$ Adler and Gottfried sum rules, for a recent discussion see also Ref. [43]. In fact, up to the additional fl_{11} contribution absent in charged-current DIS (recall Footnote 1),

$$\Delta_{\mathrm{e.m.}}\, c^{(3)}_{2,1} \;=\; \frac{d^{abc}d_{abc}}{n_c}\Big(-288+96\,\zeta_2+\frac{1472}{3}\,\zeta_3-\frac{256}{5}\,\zeta_2^2-\frac{1280}{3}\,\zeta_5\Big)$$
$$=\; -33.67693293\, n_f \qquad \text{in QCD}\;, \tag{3.5}$$

Eq. (3.4) represents the third-order coefficient-function correction to the Gottfried sum rule (GSR)[2], since the Adler sum rule involving the non-singlet coefficient function $C_{2,1}$ of the $\nu p - \bar{\nu} p$ combination does not receive any perturbative or non-perturbative corrections, see, e.g., Ref. [44].

Another interesting feature of the functions $\delta c^{(l)}_{a=2,3}$ in Eq. (2.2) is the presence of ζ-functions up to weight $2l$ in the integer moments, e.g., terms up to ζ_2^3 and ζ_3^2 occur in the third-order result (3.4). This is in contrast to the 'natural' (OPE-based) moments of $C_a^{\nu p\pm\bar{\nu}p}$ which only include contributions up to weight $2l-1$, see Refs. [28–31]. Yet the x-space expressions of all these quantities consist of harmonic polylogarithms up to weight $2l-1$ corresponding to harmonic sums up to weight $2l$. Note also that, in the approach of Refs. [20–22], the absence of weight-$2l$ terms in the natural moments appears to require a cancellation between different diagram classes.

We now return to the numerical moments (3.1) – (3.3) and investigate their consequences for the x-space functions $\delta c^{(3)}_a(x)$. We follow an approach successfully used, for instance, in Ref. [35] when only the coefficient-functions moments of Refs. [29–31] were known. Based on the two-loop end-point behaviour in Eqs. (2.7) and (2.8) we expect small-x terms up to $\ln^5 x$ and $\ln^3 x$ in $\delta c^{(3)}_{2,3}(x)$ and $\delta c^{(3)}_L(x)$, respectively, and large-x limits including contributions up to $(1-x)^{\eta_a}\ln^2(1-x)$ with $\eta_{2,3}=1$ and $\eta_L=2$. Thus the x-space expressions of $\delta c^{(3)}_a$ will be of the form

$$\delta c^{(3)}_a(x) \;=\; (1-x)^{\eta_a}\Big(\sum_{m=1}^{2} A_m \ln^m(1-x) + \delta c^{\rm smooth}_a(x) + B_1\,\frac{\ln x}{1-x}\Big) + \sum_{n=2}^{7-2\eta_a} B_n \ln^n x \tag{3.6}$$

where the functions $\delta c^{\rm smooth}_a(x)$ are finite for $0\le x\le 1$. For moment-based approximations a simple ansatz is chosen for these functions, and its free parameters are determined from the available moments together with a reasonably balanced subset of the coefficients A_m and B_n. This ansatz and the choice of the non-vanishing end-point parameters are then varied in order to estimate the remaining uncertainties of $\delta c^{(3)}_a(x)$. Finally for each value of a two (out of about 50) approximations, denoted below by A and B, are selected which indicate the widths of the uncertainty bands.

For F_2 and F_L these functions are, with $L_0=\ln x$, $x_1=1-x$ and $L_1=\ln x_1$,

$$\begin{aligned}
\delta c^{(3)}_{2,A}(x) \;&=\; (54.478\,L_1^2+304.6\,L_1+691.68\,x)\,x_1+179.14\,L_0-0.1826\,L_0^3 \\
&\quad + n_f\,\{(20.822\,x^2-282.1\,(1+\tfrac{x}{2}))\,x_1-(285.58\,x+112.3-3.587\,L_0^2)\,L_0\}\;, \\
\delta c^{(3)}_{2,B}(x) \;&=\; -(13.378\,L_1^2+97.60\,L_1+118.12\,x)\,x_1-91.196\,L_0^2-0.4644\,L_0^5 \\
&\quad + n_f\,\{(4.522\,L_1+447.88\,(1+\tfrac{x}{2}))\,x_1+(514.02\,x+147.05+7.386\,L_0)\,L_0\}
\end{aligned} \tag{3.7}$$

[2] Note that our overall normalization and expansion parameter differ from those of Ref. [34]. Consequently the corresponding GSR coefficients (3.1), (3.4) and (3.5) are larger by a factor $4^l/3$ at order α_s^l than in their notation.

and

$$\begin{aligned}
\delta c^{(3)}_{L,A}(x) &= -(495.49\,x^2+906.86)\,x_1^2-983.23\,xx_1L_0+53.706\,L_0^2+5.3059\,L_0^3 \\
&\quad +n_f\,\{(29.95\,x^3-59.087\,x^2+379.91)\,x_1^2-273.042\,xL_0^2+71.482\,x_1L_0\}\;, \\
\delta c^{(3)}_{L,B}(x) &= (78.306\,L_1+6.3838\,x)\,x_1^2+20.809\,xx_1L_0-114.47\,L_0^2-22.222\,L_0^3 \qquad (3.8)\\
&\quad +n_f\,\{(12.532\,L_1+141.99\,x^2-250.62\,x)\,x_1^2-(153.586\,x-0.6569)\,x_1L_0\}\;.
\end{aligned}$$

The corresponding results for F_3 read

$$\begin{aligned}
\delta c^{(3)}_{3,A}(x) &= (3.216\,L_1^2+44.50\,L_1-34.588)\,x_1+98.719\,L_0^2+2.6208\,L_0^5 \\
&\quad -n_f\,\{(0.186\,L_1+61.102\,(1+x))\,x_1+122.51\,xL_0-10.914\,L_0^2-2.748\,L_0^3\}\;, \\
\delta c^{(3)}_{3,B}(x) &= -(46.72\,L_1^2+267.26\,L_1+719.49\,x)\,x_1-171.98\,L_0+9.470\,L_0^3 \qquad (3.9)\\
&\quad +n_f\,\{(0.8489\,L_1+67.928\,(1+\tfrac{x}{2}))\,x_1+97.922\,xL_0-17.070\,L_0^2-3.132\,L_0^3\}\;.
\end{aligned}$$

The resulting approximations for the $\nu p-\bar{\nu}p$ odd-N coefficient functions $c^{(3)}_{2,L}(x)$ are compared in Fig. 2 to their exact counterparts [24, 25] for the even-N non-singlet structure functions. The third-order even-odd differences remain noticeable to larger values of x than at two loops, e.g., up to $x\simeq 0.3$ for F_2 and $x\simeq 0.6$ for F_L for the four-flavour case shown in the figure. The moments $N=1,\,3,\,\ldots,\,9$ constrain $\delta c^{(3)}_{2,L}(x)$ very well at $x\gtrsim 0.1$, and approximately down to $x\approx 10^{-2}$.

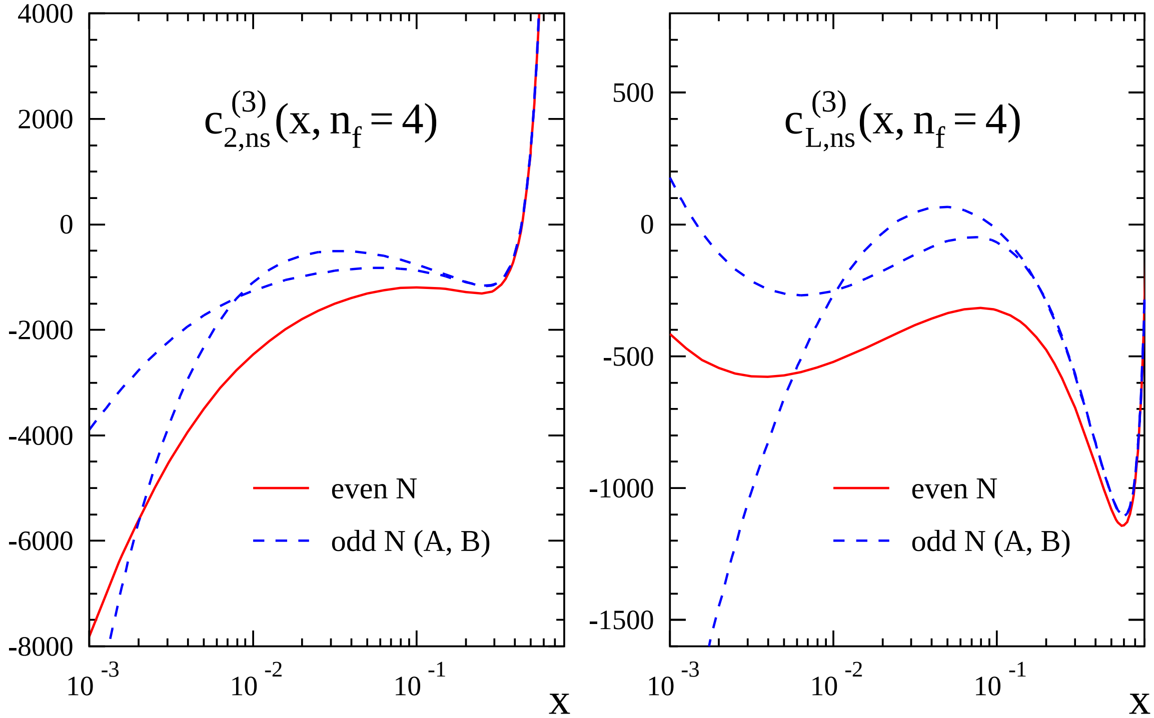

Figure 2: The exact third-order coefficient functions of the even-N structure functions $F_{2,L}^{\nu p+\bar{\nu}p}$ [24, 25] for four massless flavours, and the corresponding odd-moment quantities obtained from these results and the approximations (3.7) and (3.8) for the even – odd differences.

For some applications, such as the Paschos-Wolfenstein relation addressed in the next section, one needs the second moments of the functions $\delta c^{(3)}_{2,L}(x)$. These quantities can now be determined approximately from the above x-space results, yielding

$$\begin{aligned} \delta c^{(3)}_{2,2} &= -20.19 \pm 0.39 + (0.691 \pm 0.040)\, n_f \\ \delta c^{(3)}_{L,2} &= -24.75 \pm 0.15 - (0.792 \pm 0.014)\, n_f \; . \end{aligned} \tag{3.10}$$

Here the central values are given by the respective averages of the approximations A and B in Eqs. (3.7) and (3.8) which directly provide the upper and lower limits.

Returning to x-space we recall that uncertainty bands as in Fig. 2 do not directly indicate the range of applicability of these approximations, since the coefficient functions enter observables only via smoothening Mellin convolutions with non-perturbative initial distributions. In Fig. 3 we therefore present the convolutions of all six third-order CC coefficient functions with a characteristic reference distribution. It turns out that the approximations (3.7) and (3.8) of the previous figure can be sufficient down to values even below $x = 10^{-3}$. The uncertainty of $\delta c^{(3)}_{3}(x)$, on the other hand, becomes relevant already at larger values, $x \lesssim 10^{-2}$, as the lowest calculated moment of this quantity, $N = 2$, has far less sensitivity to the behaviour at low x.

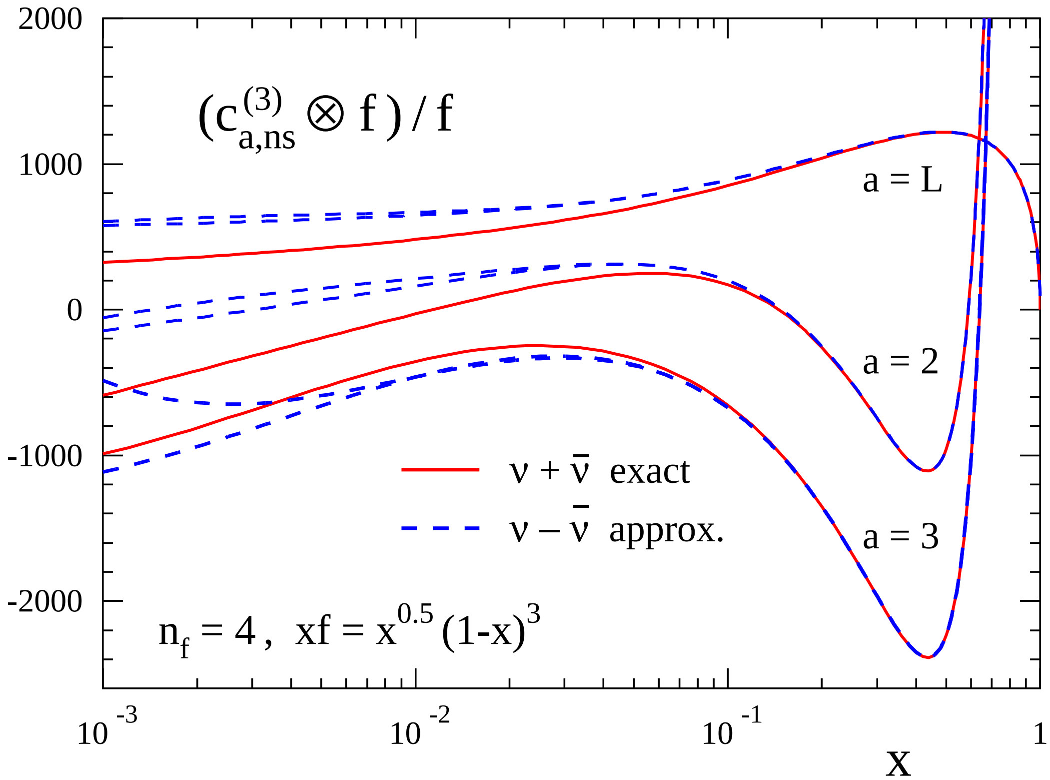

Figure 3: Convolution of the six third-order CC coefficient functions for $F_{2,3,L}$ in $\nu p + \bar{\nu} p$ [24, 25, 27] and $\nu p - \bar{\nu} p$ [Eqs. (3.7) – (3.9)] DIS with a schematic but typical non-singlet distribution f. All results have been normalized to $f(x)$, suppressing a large but trivial variation of the absolute convolutions for small and large values of x.

The three-loop corrections to the non-singlet structure functions are rather small even well below the x-values shown in the figure – recall our small expansion parameter a_s: the third-order coefficient are smaller by a factor $2.0 \cdot 10^{-3}$ if the expansion is written in powers of α_s. Their sharp rise for $x \to 1$ is understood in terms of soft-gluon effects which can be effectively resummed, if required, to next-to-next-to-next-to-leading logarithmic accuracy [45]. Our even-odd differences $\delta c_a^{(3)}(x)$, on the other hand, are irrelevant at $x > 0.1$ but have a sizeable impact at smaller x in particular on the corrections for F_2 and F_L.

4 Applications

The approximate results for $\delta c_a^{(3)}(x)$ facilitate a first assessment of the perturbative stability of the even-odd differences (2.1). In Fig. 4 we illustrate the known two orders for F_2 and F_L for $\alpha_s = 0.25$ and $n_f = 4$ massless quark flavours, employing the same reference quark distribution as in Fig. 3. Obviously our new α_s^3 corrections are important wherever these coefficient-function differences are non-negligible. On the other hand, our results confirm that these quantities are very small, and thus relevant only when a high accuracy is required. Presently this condition is fulfilled only for the determination of the weak mixing angle θ_W from neutrino DIS to which we therefore turn now.

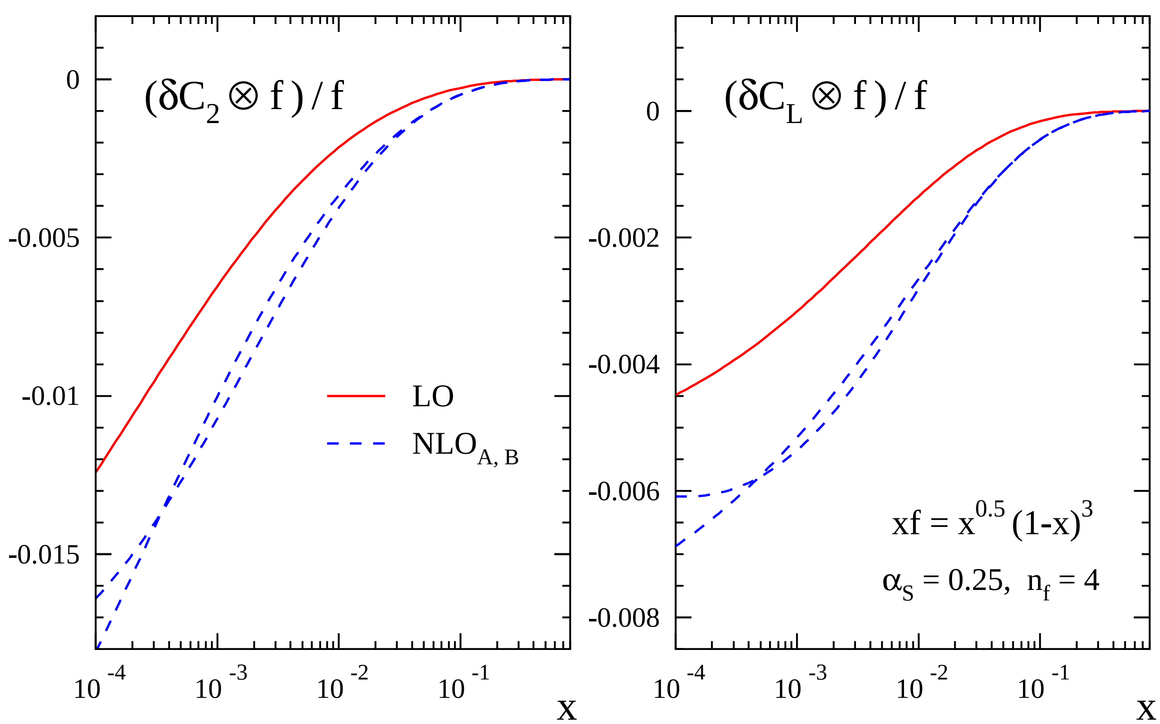

Figure 4: The first two approximations, denoted by LO and NLO, of the differences (2.2) for F_2 and F_L in charged-current DIS. The results are shown for representative values of α_s and n_f after convolution with the reference distribution $f(x)$ also employed in Fig. 3. The dashed curves correspond to the two approximations in Eqs. (3.7) and (3.8) for the new α_s^3 contributions.

For this purpose one considers the so-called Paschos-Wolfenstein relation defined in terms of a ratio of neutral-current and charged-current cross sections for neutrino-nucleon DIS [36],

$$R^- \; = \; \frac{\sigma(\nu_\mu N \to \nu_\mu X) \, - \, \sigma(\bar{\nu}_\mu N \to \bar{\nu}_\mu X)}{\sigma(\nu_\mu N \to \mu^- X) \, - \, \sigma(\bar{\nu}_\mu N \to \mu^+ X)} \; . \tag{4.1}$$

R^- directly measures $\sin^2\theta_W$ if the up and down valence quarks in the target carry equal momenta, and if the strange and heavy-quark sea distributions are charge symmetric. At the lowest order of perturbative QCD one generally finds

$$R^-_{\rm LO} \; = \; \frac{1}{2} - \sin^2\theta_W \; . \tag{4.2}$$

The quantity (4.1) has attracted considerable attention in recent years due to a determination of $\sin^2\theta_W$ by the NuTeV collaboration [11]: within the Standard Model their result is at variance with other measurements of this quantity [1], see also Refs. [13–15] for detailed discussions.

Beyond the leading order Eq. (4.2) receives perturbative QCD corrections which involve the second moments of coefficient functions for the $\nu N - \bar{\nu} N$ neutral- and charged-current structure functions.[3] Armed with the results of Sections 2 and 3 we are now able to finalize the corresponding α_s^2 contribution for massless quarks [14] and to present an accurate numerical result at order α_s^3. We denote by $q^- \equiv q - \bar{q}$ the second Mellin moments of the valence distributions of the flavours $q = u,\, d,\, s,\, \ldots,$

$$q^- \; = \; \int_0^1 dx\, x\, (q(x) - \bar{q}(x)) \; . \tag{4.3}$$

The QCD corrections to R^- can be expanded in inverse powers of the dominant isoscalar combination $u^- + d^-$ of the parton distributions – recall that the measurements of this ratio are performed for (almost) isoscalar targets. After inserting the expansion of the $\overline{\rm MS}$ coefficient functions in powers of α_s, the Paschos–Wolfenstein ratio Eq. (4.1) can be written as

$$\begin{aligned} R^- \; = \; & g_L^2 - g_R^2 + \frac{u^- - d^- + c^- - s^-}{u^- + d^-} \Bigg(3(g_{Lu}^2 - g_{Ru}^2) + (g_{Ld}^2 - g_{Rd}^2) \\ & + (g_L^2 - g_R^2) \Bigg\{ \frac{8}{9}\frac{\alpha_s}{\pi} \; + \; \frac{\alpha_s^2}{\pi^2} \left[\frac{15127}{1944} - \frac{89}{81}\zeta_2 + \frac{61}{27}\zeta_3 - \frac{32}{45}\zeta_2^2 - \frac{83}{162} n_f \right] \\ & + \; \frac{\alpha_s^3}{\pi^3} \Bigg[\frac{5175965}{52488} - \frac{356}{729}\zeta_2 - \frac{586}{27}\zeta_3 - \frac{128}{405}\zeta_2^2 + \frac{190}{81}\zeta_5 - \frac{9062}{729} n_f + \frac{2}{3} n_f \zeta_3 \\ & + \frac{226}{729} n_f^2 - \frac{1}{32}\delta c_{2,2}^{(3)} + \frac{1}{128}\delta c_{L,2}^{(3)} \Bigg] \Bigg\} \Bigg) \; + \; O\left((u^- + d^-)^{-2} \right) \; + \; O(\alpha_s^4) \; . \end{aligned} \tag{4.4}$$

Here the left- and right-handed weak couplings g_{Lu}, g_{Ld}, g_{Ru} and g_{Rd} are related to the weak mixing angle $\sin^2\theta_W$ by

$$g_L^2 \; \equiv \; g_{Lu}^2 + g_{Ld}^2 \; = \; \frac{1}{2} - \sin^2\theta_W + \frac{5}{9}\sin^4\theta_W \; , \qquad g_R^2 \; \equiv \; g_{Ru}^2 + g_{Rd}^2 \; = \; \frac{5}{9}\sin^4\theta_W \; . \tag{4.5}$$

[3] Specifically the ratio R^- includes, besides all $\nu N - \bar{\nu} N$ CC coefficient functions, the neutral-current quantity $C_3^{\rm NC}$ which is equal to its charged-current counterpart $C_3^{\nu N - \bar{\nu} N}$ at the perturbative orders considered here.

Beyond the tree level, of course, these relations receive electroweak radiative corrections, see, e.g., Ref. [46]. Eq. (4.4) shows the well-known fact that the relation (4.2) receives corrections if the parton content of the target includes an isotriplet component, $u^- \neq d^-$, or a quark sea with a C-odd component, $s^- \neq 0$ or $c^- \neq 0$. Notice also that perturbative QCD only affects these corrections.

The exact second-order contribution in Eq. (4.4) differs from the result in Ref. [14] where the function $\delta c_L^{(2)}(x)$ of Eq. (2.5) was not included. The third-order corrections can now be completed in a numerical form, using our approximations (3.10) for the second moments of $\delta c_{2,L}^{(3)}(x)$. For $n_f = 4$ flavours (and disregarding electroweak corrections) we obtain

$$\begin{aligned} R^- \;=\; & \frac{1}{2} - \sin^2\theta_W \;+\; \frac{u^- - d^- + c^- - s^-}{u^- + d^-} \left\{ 1 - \frac{7}{3}\sin^2\theta_W \;+\; \left(\frac{1}{2} - \sin^2\theta_W\right) \cdot \right. \\ & \left. \frac{8}{9}\frac{\alpha_s}{\pi}\left[1 + 1.689\,\alpha_s + (3.661 \pm 0.002)\,\alpha_s^2\right] \right\} \;+\; O\left((u^- + d^-)^{-2}\right) \;+\; O(\alpha_s^4) \;. \end{aligned} \tag{4.6}$$

The perturbation series in the square brackets appears reasonably well convergent for relevant values of the strong coupling constant, with the known terms reading, e.g., 1 + 0.42 + 0.23 for $\alpha_s = 0.25$. Thus the α_s^2 and α_s^3 contributions correct the NLO estimate by 65% in this case. On the other hand, due to the small prefactor of this expansion, the new third-order term increases the complete curved bracket in Eq. (4.5) by only about 1%, which can therefore by considered as the new uncertainty of this quantity due to the truncation of the perturbative expansion. Consequently previous NLO estimates of the effect of, for instance, the (presumably mainly non-perturbative, see Refs. [47–49]) charge asymmetry of the strange sea remain practically unaffected by higher-order corrections to the coefficient functions.

5 Summary

In this article we have presented new results for the coefficient functions of inclusive charged-current DIS in the framework of massless perturbative QCD. We have filled a gap in the two-loop literature by writing down the corresponding difference $\delta c_L^{(2)}(x)$ of the $\nu p + \bar{\nu} p$ and $\nu p - \bar{\nu} p$ structure functions F_L. Our main results are the lowest five (even- or odd-integer) Mellin moments of the third-order corrections $\delta c_a^{(3)}(x)$ for all three structure functions $F_{a=2,3,L}$ and approximations in Bjorken-x space based on these moments which are applicable down to at least $x \lesssim 10^{-2}$. As a byproduct we have calculated the related third-order coefficient-function correction to the Gottfried sum rule in photon-exchange DIS.

All our third-order results are proportional to the 'non-planar' colour factor $C_A - 2C_F$, thus confirming a conjecture by Broadhurst, Kataev and Maxwell on the $1/N_c^2$ suppression of these coefficient-function differences in the limit of a large number of colours N_c. Numerically our α_s^3 corrections prove relevant in particular for F_2 and F_L wherever the differences of the $\nu p + \bar{\nu} p$ and $\nu p - \bar{\nu} p$ coefficient functions are not negligible. We have employed the above results to derive the second- and third-order QCD corrections to the Paschos-Wolfenstein ratio R^- used to determine

the weak mixing angle from neutrino deep-inelastic scattering. The uncertainty due to uncalculated higher-order coefficient functions has been reduced to a level amply sufficient for the foreseeable future, i.e., 1% for the coefficient-function factor multiplying the quark-distribution asymmetries.

FORM files and FORTRAN subroutines with our results can be obtained from the preprint server `http://arXiv.org` by downloading the source of this article. Furthermore they are available from the authors upon request.

Note added

While this article was finalized, the 11-th moments of the functions $\delta c_2^{(3)}(x)$ and $\delta c_L^{(3)}(x)$ have been computed [50]. Both results fall into the bands generated by the respective x-space approximations in Section 3, thus confirming the reliability of these uncertainty estimates.

Acknowledgements

We would like to thank S. Alekhin, D. Broadhurst and A. Kataev for stimulating discussions. The work of S.M. and M.R. has been supported by the Helmholtz Gemeinschaft under contract VH-NG-105 and in part by the Deutsche Forschungsgemeinschaft in Sonderforschungsbereich/Transregio 9. During the final stage of this research A.V. enjoyed the hospitality of the Instituut-Lorentz of Leiden University.

Appendix

Here we present the analytic expressions for the Mellin-space coefficient-function differences $\delta c_{a,N}^{(3)}$ which were given numerically in Eqs. (3.1) – (3.3). We use the notations and conventions as specified at the beginning of Section 2 and above Eq. (2.4).

The first moment of $\delta c_2^{(3)}(x)$ has been written down in Eq. (3.4) above. The remaining known moments of this quantity are given by

$$
\begin{aligned}
\delta c_{2,3}^{(3)} \;=\;& C_F C_{FA}^2 \Big(\frac{1805677051}{466560} - \frac{2648}{9}\zeta_5 + \frac{10093427}{810}\zeta_3 - \frac{1472}{3}\zeta_3^2 - \frac{7787113}{1944}\zeta_2 \\
& + \frac{55336}{9}\zeta_2\zeta_3 - \frac{378838}{45}\zeta_2^2 - \frac{8992}{63}\zeta_2^3 \Big) \\
& + C_F^2 C_{FA} \Big(- \frac{5165481803}{1399680} + \frac{40648}{9}\zeta_5 - \frac{9321697}{810}\zeta_3 + \frac{1456}{3}\zeta_3^2 + \frac{8046059}{1944}\zeta_2 \\
& - 4984\zeta_2\zeta_3 + \frac{798328}{135}\zeta_2^2 - \frac{56432}{315}\zeta_2^3 \Big) \\
& + n_f C_F C_{FA} \Big(\frac{20396669}{116640} - \frac{1792}{9}\zeta_5 + \frac{405586}{405}\zeta_3 - \frac{139573}{486}\zeta_2 + \frac{1408}{9}\zeta_2\zeta_3 - \frac{50392}{135}\zeta_2^2 \Big) ,
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\delta c_{2,5}^{(3)} = \; & C_F C_{FA}^2 \Bigg(\frac{18473631996593}{3827250000} - \frac{17584}{45}\zeta_5 + \frac{149815672}{7875}\zeta_3 - \frac{1472}{3}\zeta_3^2 \\
& - \frac{291199027}{50625}\zeta_2 + \frac{330416}{45}\zeta_2\zeta_3 - \frac{2577928}{225}\zeta_2^2 - \frac{8992}{63}\zeta_2^3 \Bigg) \\
& + C_F^2 C_{FA} \Bigg(-\frac{16016244428419}{3827250000} + \frac{47560}{9}\zeta_5 - \frac{1270840912}{70875}\zeta_3 + \frac{1456}{3}\zeta_3^2 \\
& + \frac{1321405949}{202500}\zeta_2 - \frac{89128}{15}\zeta_2\zeta_3 + \frac{26658224}{3375}\zeta_2^2 - \frac{56432}{315}\zeta_2^3 \Bigg) \\
& + n_f C_F C_{FA} \Bigg(\frac{181199822513}{765450000} - \frac{1792}{9}\zeta_5 + \frac{6514448}{4725}\zeta_3 - \frac{1652773}{3375}\zeta_2 \\
& + \frac{1408}{9}\zeta_2\zeta_3 - \frac{11888}{27}\zeta_2^2 \Bigg) , \qquad \text{(A.2)}
\end{aligned}
$$

$$
\begin{aligned}
\delta c_{2,7}^{(3)} = \; & C_F C_{FA}^2 \Bigg(\frac{177036089007294328733}{32934190464000000} - \frac{27248}{63}\zeta_5 + \frac{65397081433}{2646000}\zeta_3 \\
& - \frac{1472}{3}\zeta_3^2 - \frac{340303364748629}{46675440000}\zeta_2 + \frac{2563996}{315}\zeta_2\zeta_3 \\
& - \frac{4570738447}{330750}\zeta_2^2 - \frac{8992}{63}\zeta_2^3 \Bigg) \\
& + C_F^2 C_{FA} \Bigg(-\frac{213694072871074531}{45177216000000} + \frac{1821772}{315}\zeta_5 - \frac{438487320707}{18522000}\zeta_3 \\
& + \frac{1456}{3}\zeta_3^2 + \frac{418808510000479}{46675440000}\zeta_2 - \frac{2071492}{315}\zeta_2\zeta_3 \\
& + \frac{6241478743}{661500}\zeta_2^2 - \frac{56432}{315}\zeta_2^3 \Bigg) \\
& + n_f C_F C_{FA} \Bigg(\frac{38079608000704561}{117622108800000} - \frac{1792}{9}\zeta_5 + \frac{22115039}{13230}\zeta_3 \\
& - \frac{113587875043}{166698000}\zeta_2 + \frac{1408}{9}\zeta_2\zeta_3 - \frac{2296328}{4725}\zeta_2^2 \Bigg) , \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
\delta c_{2,9}^{(3)} = \; & C_F C_{FA}^2 \Bigg(\frac{567651546074437032160 3}{1000376035344000000} - \frac{25664}{63}\zeta_5 + \frac{11165079556403}{375070500}\zeta_3 \\
& - \frac{1472}{3}\zeta_3^2 - \frac{8178803099431493}{945177660000}\zeta_2 + \frac{1648352}{189}\zeta_2\zeta_3 \\
& - \frac{23488033336}{1488375}\zeta_2^2 - \frac{8992}{63}\zeta_2^3 \Bigg) \\
& + C_F^2 C_{FA} \Bigg(-\frac{321022876739723700209 89}{6002256212064000000} + \frac{1162796}{189}\zeta_5 - \frac{89153747611}{3087000}\zeta_3 \\
& + \frac{1456}{3}\zeta_3^2 + \frac{342078312478997}{30005640000}\zeta_2 - \frac{1332820}{189}\zeta_2\zeta_3 \\
& + \frac{3187232017}{297675}\zeta_2^2 - \frac{56432}{315}\zeta_2^3 \Bigg)
\end{aligned}
$$

$$+n_f C_F C_{FA}\left(\frac{21832132134852204299}{52400649470400000}-\frac{1792}{9}\zeta_5+\frac{6271692134}{3274425}\zeta_3 -\frac{1931824297943}{2250423000}\zeta_2+\frac{1408}{9}\zeta_2\zeta_3-\frac{164116}{315}\zeta_2^2\right). \tag{A.4}$$

The corresponding lowest five odd-integer moments for the longitudinal structure function read

$$\begin{aligned}
\delta c_{L,1}^{(3)} &= C_F C_{FA}^2\left(\frac{21977}{9}-\frac{608}{3}\zeta_5-\frac{2648}{9}\zeta_3-\frac{3068}{9}\zeta_2-448\zeta_2\zeta_3-336\zeta_2^2\right)\\
&\quad +C_F^2 C_{FA}\left(-\frac{17819}{9}-\frac{1568}{3}\zeta_5+\frac{5648}{9}\zeta_3+\frac{1376}{9}\zeta_2+288\zeta_2\zeta_3+\frac{2304}{5}\zeta_2^2\right)\\
&\quad +n_f C_F C_{FA}\left(\frac{1366}{9}-\frac{496}{9}\zeta_3-\frac{328}{9}\zeta_2-\frac{224}{15}\zeta_2^2\right), \qquad (A.5)
\end{aligned}$$

$$\begin{aligned}
\delta c_{L,3}^{(3)} &= C_F C_{FA}^2\left(-\frac{12350749}{19440}+352\zeta_5+\frac{52516}{45}\zeta_3+\frac{47}{27}\zeta_2+96\zeta_2\zeta_3-\frac{7544}{15}\zeta_2^2\right)\\
&\quad +C_F^2 C_{FA}\left(\frac{10152961}{12960}-368\zeta_5-\frac{16412}{15}\zeta_3-242\zeta_2+144\zeta_2\zeta_3+\frac{1168}{3}\zeta_2^2\right)\\
&\quad +n_f C_F C_{FA}\left(-\frac{16757}{1620}+\frac{2936}{45}\zeta_3-\frac{16}{9}\zeta_2-\frac{368}{15}\zeta_2^2\right), \qquad (A.6)
\end{aligned}$$

$$\begin{aligned}
\delta c_{L,5}^{(3)} &= C_F C_{FA}^2\left(-\frac{735306721}{17010000}-\frac{1888}{3}\zeta_5+\frac{558244}{315}\zeta_3-\frac{442783}{675}\zeta_2\right.\\
&\qquad\qquad \left.+448\zeta_2\zeta_3-\frac{4160}{9}\zeta_2^2\right)\\
&\quad +C_F^2 C_{FA}\left(\frac{6741265367}{10206000}-\frac{736}{3}\zeta_5-\frac{1285168}{945}\zeta_3+\frac{51493}{405}\zeta_2\right.\\
&\qquad\qquad \left.+96\zeta_2\zeta_3+\frac{69608}{225}\zeta_2^2\right)\\
&\quad +n_f C_F C_{FA}\left(-\frac{2107157}{255150}+\frac{8816}{105}\zeta_3-\frac{11992}{405}\zeta_2-\frac{736}{45}\zeta_2^2\right), \qquad (A.7)
\end{aligned}$$

$$\begin{aligned}
\delta c_{L,7}^{(3)} &= C_F C_{FA}^2\left(\frac{354522585410107}{666792000000}-1408\zeta_5+\frac{47266403}{23625}\zeta_3-\frac{1095179473}{945000}\zeta_2\right.\\
&\qquad\qquad \left.+752\zeta_2\zeta_3-\frac{147056}{375}\zeta_2^2\right)\\
&\quad +C_F^2 C_{FA}\left(\frac{11388456807174161}{28005264000000}-184\zeta_5-\frac{176925641}{132300}\zeta_3+\frac{4569363329}{13230000}\zeta_2\right.\\
&\qquad\qquad \left.+72\zeta_2\zeta_3+\frac{663878}{2625}\zeta_2^2\right)\\
&\quad +n_f C_F C_{FA}\left(\frac{369546282989}{50009400000}+\frac{124282}{1575}\zeta_3-\frac{220747}{5250}\zeta_2-\frac{184}{15}\zeta_2^2\right), \qquad (A.8)
\end{aligned}$$

$$
\begin{aligned}
\delta c_{L,9}^{(3)} &= C_F C_{FA}^2\left(\frac{1346454911003496947}{1323248724000000}-\frac{10528}{5}\zeta_5+\frac{13247918}{6125}\zeta_3\right.\\
&\quad\left.-\frac{325373958827}{208372500}\zeta_2+\frac{5184}{5}\zeta_2\zeta_3-\frac{296736}{875}\zeta_2^2\right)\\
&\quad+C_F^2C_{FA}\left(\frac{17693872049573089}{73513818000000}-\frac{736}{5}\zeta_5-\frac{125991917}{99225}\zeta_3\right.\\
&\quad\left.+\frac{10496201057}{23152500}\zeta_2+\frac{288}{5}\zeta_2\zeta_3+\frac{1688888}{7875}\zeta_2^2\right)\\
&\quad+n_fC_FC_{FA}\left(\frac{23852323249607}{1444021425000}+\frac{1249264}{17325}\zeta_3-\frac{1542176}{33075}\zeta_2-\frac{736}{75}\zeta_2^2\right). \qquad \text{(A.9)}
\end{aligned}
$$

Finally the analytic expressions for the moments $\delta c_{2,N}^{(3)}$ in Eq. (3.3) are

$$
\begin{aligned}
\delta c_{3,2}^{(3)} &= C_F C_{FA}^2\left(-\frac{840949}{243}+\frac{9344}{9}\zeta_5-\frac{650360}{81}\zeta_3+\frac{1472}{3}\zeta_3^2+\frac{712328}{243}\zeta_2\right.\\
&\quad\left.-\frac{47920}{9}\zeta_2\zeta_3+\frac{30416}{5}\zeta_2^2+\frac{8992}{63}\zeta_2^3\right)\\
&\quad+C_F^2C_{FA}\left(\frac{15979879}{4374}-\frac{38416}{9}\zeta_5+\frac{580504}{81}\zeta_3-\frac{1456}{3}\zeta_3^2-\frac{742390}{243}\zeta_2\right.\\
&\quad\left.+4368\zeta_2\zeta_3-\frac{577264}{135}\zeta_2^2+\frac{56432}{315}\zeta_2^3\right) \qquad \text{(A.10)}\\
&\quad+n_fC_FC_{FA}\left(-\frac{119522}{729}+\frac{1792}{9}\zeta_5-\frac{57128}{81}\zeta_3+\frac{46112}{243}\zeta_2-\frac{1408}{9}\zeta_2\zeta_3+\frac{40024}{135}\zeta_2^2\right),\\
\delta c_{3,4}^{(3)} &= C_F C_{FA}^2\left(-\frac{21230721185377}{4374000000}+\frac{23704}{45}\zeta_5-\frac{292322783}{20250}\zeta_3+\frac{1472}{3}\zeta_3^2\right.\\
&\quad\left.+\frac{5644168873}{1215000}\zeta_2-\frac{100792}{15}\zeta_2\zeta_3+\frac{6477802}{675}\zeta_2^2+\frac{8992}{63}\zeta_2^3\right)\\
&\quad+C_F^2C_{FA}\left(\frac{19991706724601}{4374000000}-5208\zeta_5+\frac{272933467}{20250}\zeta_3-\frac{1456}{3}\zeta_3^2\right.\\
&\quad\left.-\frac{6307524619}{1215000}\zeta_2+\frac{253064}{45}\zeta_2\zeta_3-\frac{2500616}{375}\zeta_2^2+\frac{56432}{315}\zeta_2^3\right)\\
&\quad+n_fC_FC_{FA}\left(-\frac{15339664501}{72900000}+\frac{1792}{9}\zeta_5-\frac{755894}{675}\zeta_3+\frac{21942049}{60750}\zeta_2\right.\\
&\quad\left.-\frac{1408}{9}\zeta_2\zeta_3+\frac{53128}{135}\zeta_2^2\right), \qquad \text{(A.11)}\\
\delta c_{3,6}^{(3)} &= C_F C_{FA}^2\left(-\frac{172761364527374293}{32162295375000}+\frac{21200}{63}\zeta_5-\frac{3380925064}{165375}\zeta_3+\frac{1472}{3}\zeta_3^2\right.\\
&\quad\left.+\frac{147865501939}{24310125}\zeta_2-\frac{2395856}{315}\zeta_2\zeta_3+\frac{75351016}{6125}\zeta_2^2+\frac{8992}{63}\zeta_2^3\right)\\
&\quad+C_F^2C_{FA}\left(\frac{313157547783370669}{64324590750000}-\frac{1810712}{315}\zeta_5+\frac{67828543996}{3472875}\zeta_3-\frac{1456}{3}\zeta_3^2\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{3604225183081}{486202500}\zeta_2+\frac{667864}{105}\zeta_2\zeta_3-\frac{1397140016}{165375}\zeta_2^2+\frac{56432}{315}\zeta_2^3\bigg) \\
&+n_f C_F C_{FA}\bigg(-\frac{503591542653161}{1837845450000}+\frac{1792}{9}\zeta_5-\frac{16004944}{11025}\zeta_3+\frac{23420609}{42875}\zeta_2 \\
&\quad -\frac{1408}{9}\zeta_2\zeta_3+\frac{2135888}{4725}\zeta_2^2\bigg)\,, \qquad \text{(A.12)}
\end{aligned}
$$

$$
\begin{aligned}
\delta c_{3,8}^{(3)} \;=\;& C_F C_{FA}^2\bigg(-\frac{4588277528647792706731 1}{8003008282752000000}+\frac{22640}{63}\zeta_5-\frac{38829577931303}{1500282000}\zeta_3 \\
&\quad +\frac{1472}{3}\zeta_3^2+\frac{5677110453154657}{756142128000}\zeta_2-\frac{7858468}{945}\zeta_2\zeta_3 \\
&\quad +\frac{43146817871}{2976750}\zeta_2^2+\frac{8992}{63}\zeta_2^3\bigg) \\
&+C_F^2 C_{FA}\bigg(\frac{126830527574348410837327}{24009024848256000000}-\frac{5795836}{945}\zeta_5+\frac{4175977929883}{166698000}\zeta_3 \\
&\quad -\frac{1456}{3}\zeta_3^2-\frac{194854342276579}{20003760000}\zeta_2+\frac{6509876}{945}\zeta_2\zeta_3 \\
&\quad -\frac{58805551031}{5953500}\zeta_2^2+\frac{56432}{315}\zeta_2^3\bigg) \\
&+n_f C_F C_{FA}\bigg(-\frac{3380190329263337489}{9527390812800000}+\frac{1792}{9}\zeta_5-\frac{1026540911}{595350}\zeta_3 \\
&\quad +\frac{3263620615369}{4500846000}\zeta_2-\frac{1408}{9}\zeta_2\zeta_3+\frac{778496}{1575}\zeta_2^2\bigg)\,, \qquad \text{(A.13)}
\end{aligned}
$$

$$
\begin{aligned}
\delta c_{3,10}^{(3)} \;=\;& C_F C_{FA}^2\bigg(-\frac{2924815993615556996346598663}{483334682604559632000000}+\frac{210944}{385}\zeta_5 \\
&\quad -\frac{3080312718428437}{99843767100}\zeta_3+\frac{1472}{3}\zeta_3^2+\frac{13720175530646448109}{1537594013340000}\zeta_2 \\
&\quad -\frac{8455904}{945}\zeta_2\zeta_3+\frac{3568998808}{218295}\zeta_2^2+\frac{8992}{63}\zeta_2^3\bigg) \\
&+C_F^2 C_{FA}\bigg(\frac{6191658137399697511925144182 1}{1063336301730031190400000 0}-\frac{66873844}{10395}\zeta_5 \\
&\quad +\frac{30055925797598243}{998437671000}\zeta_3-\frac{1456}{3}\zeta_3^2-\frac{318659860647520101 1}{263587545144000}\zeta_2 \\
&\quad +\frac{75900812}{10395}\zeta_2\zeta_3-\frac{1995571648453}{180093375}\zeta_2^2+\frac{56432}{315}\zeta_2^3\bigg) \\
&+n_f C_F C_{FA}\bigg(-\frac{339629926756418877268603}{767197908896126400000}+\frac{1792}{9}\zeta_5-\frac{70469642338}{36018675}\zeta_3 \\
&\quad +\frac{2677118231310293}{2995313013000}\zeta_2-\frac{1408}{9}\zeta_2\zeta_3+\frac{1015276}{1925}\zeta_2^2\bigg)\,. \qquad \text{(A.14)}
\end{aligned}
$$

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